

A CONTEMPORARY APPROACH TO THE PROBLEM OF DETERMINING
PHYSICAL PARAMETERS ACCORDING TO THE RESULTS OF MEASUREMENTS

P. Ye. El'yasberg

(NASA-TM-75616) A CONTEMPORARY APPROACH TO
THE PROBLEM OF DETERMINING PHYSICAL
PARAMETERS ACCORDING TO THE RESULTS OF
MEASUREMENTS. (National Aeronautics and Space
Administration) 53 p HC A04/MF A01 CSCI 12A G3/64

N79-24696

Unclassified
20930

Translation of "Sovremennyj podkhod k zadache opredeleniya
fizicheskikh parametrov po rezul'tatam izmerenij", Academy of
Sciences of the USSR, Institute of Space Research, Moscow,
Report Pr-429, 1978, pp. 1-55.



NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546

MAY 1979

STANDARD TITLE PAGE

1. Report No NASA TM-75616	2. Government Accession No.	3. Recipient's Catalog No	
4. Title and Subtitle A CONTEMPORARY APPROACH TO THE PROBLEM OF DETERMINING PHYSICAL PARAMETERS ACCORDING TO THE RESULTS OF MEASUREMENTS		5. Report Date May 1979	6. Performing Organization Code
7. Author(s) P. Ye. El'yasberg		8. Performing Organization Report No	
9. Performing Organization Name and Address Leo Kanner Associates Redwood City, California 94063		10. Work Unit No	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration, Washington, D.C. 20546		11. Contract or Grant No NASW-3199	
15. Supplementary Notes Translation of "Sovremennyj podkhod k zadache opredeleniya fizicheskikh parametrov po rezul'tatam izmereniy", Academy of Sciences of the USSR, Institute of Space Research, Moscow, Report Pr-429, 1978, pp. 1-55.		13. Type of Report and Period Covered Translation	
16. Abstract The report compares different approaches to the problem of determining physical parameters, according to the results of measurements. The shortcomings of the classical approach are set forth, and the newer methods resulting from these shortcomings are explained. The problem is approached with the assumption that the probabilities of error are known, as well as without knowledge of the distribution of the probabilities of error. In general, the report illustrates how this newer approach avoids the errors and shortcomings inherent in the classical approach.		14. Sponsoring Agency Code	
17. Key Words (Selected by Author(s))		18. Distribution Statement Unclassified-Unlimited	
19. Security Classif. (of this report) Unclassified	20. Security Classif. (of this page) Unclassified	21. No. of Pages	22. Price

A CONTEMPORARY APPROACH TO THE PROBLEM OF DETERMINING
PHYSICAL PARAMETERS ACCORDING TO THE RESULTS OF MEASUREMENTS

P. Ye. El'yasberg

Compared in the study are different approaches to problems of determining (evaluating) some physical (biological, economic, etc.) parameters according to the results of measurements. Shown in the simplest examples are the shortcomings of the classical approach, which gives rise to the method of maximum probability and its various modifications (method of least squares, optimal filtration, etc.), with which the distribution of probabilities of errors of the initial data is assumed as given. Also examined is the approach to the problems of evaluation, used in recent years, when the distribution of errors is considered unknown, and only some set, to which this distribution belongs, is given. /2*

The basic results, obtained with this assumption, are described. It is shown using examples that these results are devoid, to a considerable extent, of the shortcomings inherent in the classical approach, and this approach is closer, in its nature, to the practice of solving problems of evaluation.

1. Posing of the Problem /3

With the processing of great masses of information (during processing of data of space and land physical experiments, in geodesy, during the solution of a number of biological problems, in economics, during the determination of the initial data for controlling the movement of various systems, etc.), it is often necessary to solve problems of

*Numbers in the margin indicate pagination in the foreign text.

determining some physical (biological, economic, etc.) parameters according to the results of measurements.

We will use $q=\{q_1, \dots, q_m\}$ to designate the vector of the parameters which are to be determined, and $d=\{d_1, \dots, d_m\}$ to designate the vector of the measured magnitudes. There occurs the relationship $n \geq m$ between the number m of the parameters being determined and the number n of the magnitudes being measured. With $n=m$, it can be said that the problem is solved according to a sufficient number of measurements, and with $n > m$, according to a surplus number of measurements. We will assume that there is some mathematical model of the phenomenon in question, which makes it possible to construct a dependence of the type $d=d(q)$. Then, the following dependence exists between the measured value of \tilde{d} of the vector d and the vector q :

$$\tilde{d} = d(q) + \xi$$

(1)

14

where $\xi=\{\xi_1, \dots, \xi_n\}$ is the vector of the total errors of measurement in the model $d=d(q)$.

With these designations, the problem amounts to determining the so-called evaluation \hat{q} of the vector q , according to measurements of d using some algorithm of evaluation $\hat{q}=q(\tilde{d})$. The purpose of this algorithm is the potential elimination of the effect of the errors ξ . With a surplus number of measurements ($n > m$), the problem of constructing an algorithm of evaluation, generally speaking, is not similar. Therefore, the problem occurs of selecting an optimal algorithm. Different criteria of optimality may be examined (rapidity of calculation, simplicity of algorithm, etc.). We will subsequently proceed from the conditions of achieving maximum accuracy and

stability of evaluation of \hat{q} .

Along with the problem of finding the magnitude of $\hat{q}=\hat{q}(\tilde{d})$, it is of great significance to evaluate the accuracy of the obtained result. This is important, for example, during the comparison of results obtained on the basis of different mathematical models, and according to different measurements, as well as in those cases when the found magnitude of \hat{q} is utilized for purposes of controlling the movement of some system (for example, for control of the flight of a spacecraft).

We will subsequently call all of the problems examined here problems of evaluation.

2. Classical Approach to Problems of Evaluation

The construction of the optimal algorithm $\hat{q}=\hat{q}(\tilde{d})$ and the evaluation of the accuracy of the obtained magnitude of q depend substantially on the adopted assumptions on the error ζ , which is part of the right side of dependence (1). We will call the set of these assumptions the model of errors in the future. 15

With the classical posing of the problem, ζ is viewed as a random vector with a density of probabilities $F(\zeta, p)$, given with an accuracy of up to some vector $p=\{p_1, \dots, p_K\}$ of the parameters of distribution. In this case, the evaluations of \hat{q} and \hat{p} are found using the so-called method of maximum probability, which consists of the search for the minimum of the function of probability $\phi(q)=F[\tilde{d}-d(q)]$. In other words, in this case

$$\hat{p}, \hat{q} = \arg \max_{p, q} F\{[\hat{d} - d(q)], p\}. \quad (2)$$

Assuming that the distribution $F(\zeta)$ of the vector of ζ is normal with a given mathematical expectancy and a co-variation matrix $D(\zeta)$, which is known with an accuracy of up to some positive multiple σ^2 , the method of maximum probability is reduced to the method of least squares. If $E(\zeta)$ is given, then one can always reduce the problem to a form in which $E(\zeta)=0$ [1]. Thus, the problem, according to the method of least squares, is solved with the assumption that

$$F(\zeta) = N(0, \sigma^2 L), \quad (3)$$

where $N[E(\zeta), D(\zeta)]$ is the known expression for the density of a multiple normal distribution with given mathematical expectancy $E(\zeta)$ and covariation matrix $D(\zeta)$, and L is the given positively-determined matrix. /6

By making use of dependences (2) and (3), we obtain an algorithm of evaluation according to the method of least squares [1].

$$\hat{q} = \arg \min_q [\tilde{d} - d(q)]^T L^{-1} [\tilde{d} - d(q)]. \quad (4)$$

We will note that, in the literature, the method of least squares is often understood as algorithm (4) in that case when the matrix L is diagonal. However, we will proceed here from the more general determination (4) of this method.

The methods of maximum probability and least squares have a long history. The method of least squares was first developed by Gauss in 1794. For nearly two centuries since this time, this method recommended itself well for the solution of various problems of celestial mechanics, geodesy, experi-

mental physics, and other sciences. The widespread dissemination of the indicated methods in the common form, as well as in various modifications (method of a maximum of empirical probability, Kalman's filter, method of optimum allocation, etc.), evoked the interest of a number of leading specialists in the area of mathematical statistics. Their labors underlay the conducting of a serious mathematical investigation of these methods, in which it was shown that they possess a number of appreciable optimal properties [2]. These properties have an asymptotic nature, i.e., they are demonstrated with a number of measurements $\rightarrow \infty$. In this case

- the mathematical expectancy of error δq of the evaluation of \hat{q} approaches zero (asymptotic empiricalness);
- the dispersions of all of the components of the error δq approach their minimum values in a great number of all the possible empirical algorithms of filtration (asymptotic effectiveness);

- the distribution of the errors of the obtained evaluations approaches normal (asymptotic normalcy);

- the evaluation of q , with some additional assumptions, coincides, in probability, with the true value of q (consistency).

We will note that the effectiveness of the method of maximum probability, strictly speaking, does not hold for all, but only for "nearly all" of the true values of the vector q . In a great number of these vectors, so-called supereffective evaluations may exist [3].

In addition, the indicated properties are usually proved with the assumption that the vector ξ may be presented in the form of a set of mutually independent and equally distributed vectors. If this condition is not fulfilled, then some of the indicated properties may not occur. Specifically, in this case, the evaluation according to the method of least squares

may be inconsistent [1,4].

The following practical conclusions are usually drawn from the indicated properties of the examined evaluations:

-any enlisting for the solution of the problem, of additional measurements is useful, since it leads to improvement of the accuracy of solution of the problem;

-by increasing the number of utilized measurements without restriction, one can obtain as high an accuracy of evaluation of \hat{q} as is desired:

-

The indicated theoretical properties of the evaluations of the maximum probability evoked in the experimenters and processors of information a striving to ~~enfist*many o moneon~~ /8 measurement se ast possiblén fört the solution of problems of the examined type. Discovered for this purpose in recent years were greater possibilities, determined by the rapid development of measuring and calculating technology. Many thousands of measurements are utilized in a number of critical problems. However, in practice, it turns out that an unrestricted increase in the number n of measurements does not provide the desired effect. Beginning with some n , the accuracy \hat{q} practically ceases to increase, often beginning to worsen. In this case, it turns out that the evaluations of the accuracy \hat{q} , obtained on the basis of the theory of the methods of maximum probability and least squares, are unjustifiably optimistic. The latter circumstance is especially intolerable during the solution of critical problems (determination of basic physical constants, comparison of results obtained on the basis of various mathematical models, determination of initial data for the control of movement of different systems, etc.).

The indicated circumstances are well-known by many re-

searchers, who are practically occupied by questions of processing great masses of information. In them, a sense of distrust has long been rooted towards the evaluation of the accuracy obtained on the basis of the theory of the methods of maximum probability and least squares. This lack of correspondence between the theory and practice may be explained only by deviations of the real conditions of the experiment from the assumptions utilized during the construction of the methods of maximum probability and least squares (since the theory of these methods is mathematically strict).

As is evident from expressions (2), (3), and (4), the algorithms of evaluation, according to the examined methods, depend substantially on the distribution $F(\zeta)$ of the errors ζ . During the construction of these algorithms, it is necessary that this distribution be given in advance. With a deviation of the actual distribution $F(\zeta)$ from the adopted distribution, the optimal properties indicated above do not occur. We will elucidate this fact in examples. 19

We will examine the problem of determining some scalar magnitude of q , according to its measured values of $\tilde{d}_i (i=1, \dots, n)$. In this case, dependence (4) takes on the form

$$\tilde{d}_i = q + \zeta_i \quad (i=1, \dots, n) \quad (5)$$

During construction of the algorithm of evaluation, we will propose that the errors $\zeta_i (i=1, \dots, n)$ are distributed normally, with zero mathematical expectancy and a dispersion σ^2 . In this case, the correlation between the different errors is absent. Then, expression (3) may be written in the form

$$F(\zeta) = N(0, \sigma^2 I), \quad (6)$$

where I is a unit matrix.

With these assumptions, algorithm (4) of evaluation, according to the method of least squares, is reduced to the search for the arithmetic mean

$$\hat{q} = \frac{1}{n} \sum_{i=1}^n \hat{d}_i. \quad (7)$$

We will designate the error in the obtained evaluation by $\delta q = \hat{q} - q$, and adopt the following magnitude in the capacity of the characteristic of accuracy

$$\beta = \sqrt{E(\delta^2 q)},$$

where $E(\delta^2 q)$ is the mathematical expectancy of the square of the error δq .

We will utilize β to designate the value of β with the utilization of algorithm (7) and assumption (6). It is easy to show that

$$\beta_0 = \frac{\sigma}{\sqrt{n}}. \quad (8)$$

From expression (8), it follows directly that

$$\lim_{n \rightarrow \infty} \beta_0 = 0, \quad (9)$$

i.e., evaluation (7), with assumption (6), is consistent. In this case, an increase in the number n of utilized measurements always leads to an increase in the accuracy of evaluation of \hat{q} .

We will now assume that assumption (6) is actually not satisfied, but we will continue to utilize algorithm (7). Let

the errors ζ_i , in actuality, be represented in the form

$$\zeta_i = \zeta'_i + \zeta''_i \quad (i=1, \dots, n), \quad (10)$$

where ζ'_i are the errors of the model $d=q$, and ζ''_i are the errors in the measurements.

The errors ζ'_i occur because of the fact that the measured magnitude of d is not constant in actuality, but changes according to some law. Suppose that the only thing that we know about this law is the fact that the rate \dot{d} of change in d is restricted according to a modulus, i.e., $|\dot{d}| \leq v$, where v is a given constant. We will adopt the value of d at the initial moment in time $t=0$ as the parameter of q being evaluated. We will assume that the measurements of d are carried out at the moments $t_i = \pm j\tau$ ($j=0, 1, \dots, \frac{n-1}{2}$), where τ is the given spacing of the change in the time of the measurements (here, we will limit ourselves to the case of unevenness of the number n of the measurements). With these assumptions

$$|\zeta'_i| \leq j\tau v \quad (j=0, 1, \dots, \frac{n-1}{2}) \quad (11)$$

11

As far as the errors ζ''_i in measurement are concerned, we will assume that their mathematical expectancies are equal to zero, the dispersions $D(\zeta''_i)$ do not exceed the given magnitude of σ_{\max}^2 , and the coefficients of correlation $K(\zeta''_i, \zeta''_e)$ ($t_i \neq t_e$) do not exceed the given K_{\max} ($|K| \leq 1$). Making use of the dependences (10) and (11), we find that the mathematical expectancies $E(\zeta_i)$, the dispersions $D(\zeta_i)$, and the coefficients of correlation $K(\zeta_i, \zeta_e)$ of the errors ζ_i satisfy the inequalities

$$|E(\zeta_i)| \leq j\tau v, D(\zeta_i) \leq \sigma_{\max}^2, |K(\zeta_i, \zeta_e)| \leq K_{\max} \quad (12)$$

Comparison of models (6) and (12) of the errors shows that the former assumes a sufficiently complete knowledge of the distribution of the errors ζ , while the latter imposes only slight restrictions on the characteristics of this distribution. Such an approach is considerably closer to the real conditions, under which applied problems of information processing are solved. In this case, it does not seem possible to determine the accurate value of the magnitude of β . However, it is possible to find its so-called guaranteed value

$$\beta_{\text{guar}} = \max_{F(\zeta) \in \mathcal{F}} \beta, \quad (13)$$

where \max appears in the set \mathcal{F} of all of the distributions which satisfy the conditions in (12).

Making use of dependences (5) and (12), one can show that during the determination of the evaluation of q , using algorithm (7)

$$\beta_{\text{guar}} = \sigma_{\max} \sqrt{\frac{1-K}{n} + K + \frac{\gamma^2}{4} (n - \frac{1}{2})^2}, \quad (14)$$

12

where $\gamma = \frac{\sigma}{\sigma_{\max}}$.

It is evident from expression (14) that, as a function of the values of the magnitudes of K and γ , β_{guar} either increases monotonously with an increase in the number n of measurements, or diminishes at first, reaches some minimum, and then begins to increase. With $\gamma=0$, the minimum is not reached for a finite n . In all cases

$$\lim_{n \rightarrow \infty} \beta_{\text{guar}} = \begin{cases} \sigma_{\max} \sqrt{K} & \text{with } \gamma=0, \\ \infty & \text{with } \gamma \neq 0. \end{cases} \quad (15)$$

Depicted by the solid line in figure 1 is the graph of the dependence of the ratio $\beta_{\text{guar}}/\sigma_{\max}$ on n with $K=0.01$, $\gamma=0.01$. The dotted line in this same figure depicts the graph of the dependence β_0/σ on n , calculated according to formula (8).

It is evident from the given results that, with the assumptions in (12), evaluation (7) proves inconsistent (one can show that, with these assumptions, it is generally impossible to construct a consistent evaluation). With $\gamma \neq 0$, there exists some optimal number n_{opt} of utilized measurements. With $n > n_{\text{opt}}$, any increase in the number n leads to impairment of the accuracy of the evaluation of q .

Thus, the switch from model (6) of the errors to model (12) fundamentally changes the nature of the dependence of the evaluation of the accuracy of the magnitude of \hat{q} on the number n of utilized measurements. In this case, model (12) ensures obtaining of results which are considerably closer to practice in the processing of measurement information. 13

From (8) and (15), it follows that if at least one of the magnitudes of K or $\gamma \neq 0$, then

$$\lim_{n \rightarrow \infty} \frac{\beta_{\text{guar}}}{\beta_0} \rightarrow \infty. \quad (16)$$

Thus, with as small a deviation from the ideal model (6) as desired, and a sufficiently great n , expression (8) does not at all characterize the actual accuracy of the evaluation (7). This phenomenon we will subsequently call the instability of the magnitude of β_0 , according to the basic assumptions. In practice, some deviations from model (6) are unavoidable. Therefore, expression (8) may be utilized only with small n . With sufficiently large n , it is necessary to switch to a model of errors which takes into account the potential spread of the characteristics of distribution of the vector ζ in prob-

lens of evaluation of the accuracy of the magnitude of \hat{q} and the selection of the optimal composition of the measurements. In this case, evaluation (7) proves to be biased and inconsistent.

In the capacity of a second example, we will examine the problem of determining the velocity v of the change in some magnitude d at the initial moment in time $t=0$. Here, we will make use of the linear model

$$\tilde{d}_i = a + vt_i + \xi_i \quad (i=1, \dots, n), \quad (17)$$

where \tilde{d}_i is the measured value of d , a is some constant, t_i is the time of the measurements, and ξ_i is the error.

We will assume that the moments t_j are determined by the expressions

$$t_j = \pm \frac{T}{j} \quad (j=1, \dots, \frac{n}{2}), \quad (18)$$

where T is the given time.

14

By solving the problem by the method of least squares, with assumption (6), we find the desired evaluation:

$$\hat{v} = \frac{\sum_{i=1}^n t_i \tilde{d}_i}{\sum_{i=1}^n t_i^2}. \quad (19)$$

The dispersion of this evaluation with assumption (6) is

$$D_0(\hat{v}) = \frac{\sum_{i=1}^n t_i^2}{\sum_{i=1}^n t_i^2} = \frac{\sigma^2}{2T^2 \sum_{j=1}^{\frac{n}{2}} \frac{1}{j^2}}. \quad (20)$$

We will note that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{i^2} = \frac{\pi^2}{6} . \quad (21)$$

From here, it follows directly that

$$\lim_{n \rightarrow \infty} D_0(\hat{v}) = \frac{5G^2}{\pi^2 T^2} . \quad (22)$$

Thus, although the magnitude of $D_0(\hat{v})$ diminishes with an increase in the number n of measurements, it has a lower boundary which is not equal to zero. Therefore, evaluation (19), with assumption (6), proves inconsistent. This is associated with the selection of the time of the measurements (18).

We will now assume that the errors ξ_i are correlated among themselves. Here, the coefficients of the correlation are

$$K(\xi_i, \xi_e) = \begin{cases} k_1 \text{ with } t_i \neq t_e, t_i, t_e > 0 \\ k_2 \text{ with } t_i, t_e < 0 \end{cases} \quad (23)$$

where k_1 and k_2 are given magnitudes, which satisfy the conditions

$$k_1 > k_2, k_1 + k_2 > 0, |k_1| \leq 1, |k_2| \leq 1. \quad (24)$$

15

One can show that the correlation matrix made up of these coefficients is positively definite, i.e., the conditions in (23) are correct.

With proposal (23), and the assumption that $D(\xi_i) = \sigma^2$ ($i=1, \dots, n$), the dispersion of evalution (19) is determined by the expression

$$D_1(\hat{v}) = \left[\frac{1-k_1}{2 \sum_{j=1}^{n/2} \frac{1}{j^2}} + \frac{k_1-k_2}{2} \left(\sum_{j=1}^{n/2} \frac{1}{j} \right)^2 \right]. \quad (25)$$

Then, making use of the inequality

$$\sum_{j=1}^{n/2} \frac{1}{j} > \ln \left(\frac{n}{2} + 1 \right), \quad (26)$$

we find that

$$D_1(v) > \frac{\sigma^2}{\pi^2} \cdot \frac{k_1-k_2}{5,5} \ln^2 \left(\frac{n}{2} + 1 \right) \quad (27)$$

From here, it follows directly that

$$\lim_{n \rightarrow \infty} D_1(\hat{v}) = \infty. \quad (28)$$

Depicted in figure 2 by the solid line is the graph of the dependence of $D_1(\hat{v}) \frac{T^2}{\sigma^2}$ on the number n , assuming that $k_1=k_2=0.1$. In the same figure, the dotted line depicts the graph of the magnitude of $D_0(\hat{v}) \frac{T^2}{\sigma^2}$, calculated according to formula (20).

Thus, in this example as well, the enlistment of additional measurements leads to an unrestricted increase in the error of the measurements. Here,

$$\lim_{n \rightarrow \infty} D_1(\hat{v}) / D_0(\hat{v}) = \infty,$$

/16

i.e., expression (20) proves unstable with any $k_1 - k_2 \neq 0$, no matter how small.

Apparent in the given examples is a sharp increase in the effect of small correlation dependences between the components of the vector ξ with an increase in the number n of these components. This may be illustrated on an example of the calculation of the dispersion of the sum of the components of the vector ξ :

$$\mathcal{D}\left(\sum_{i=1}^n \xi_i\right) = \sum_{i,j=1}^n K(\xi_i, \xi_j) \sqrt{\mathcal{D}(\xi_i) \mathcal{D}(\xi_j)} .$$

The total number of diagonal terms in the right-hand portion of this dependence is equal to n , and the number of nondiagonal terms, associated with the presence of correlation connections between the components of the vector, is equal to $-n(n-1)$. Therefore, the coefficients $K(\xi_i, \xi_j)$ of the correlation on the order of $1/n$ may have a substantial effect on the examined expression. With $K(\xi_i, \xi_j)$ on the order of n^{-2} ($0 < \alpha < 1$), their effect becomes decisive.

The instability of the classical evaluations, of accuracy, shown in the examples, has a general nature. One can prove that, if the distribution $F(\xi)$ of the errors ξ is selected so that the corresponding evaluation of the least squares (maximum probability) is theoretically consistent, then, with sufficiently general assumptions, the classical evaluation of accuracy of the obtained results proves to be unstable in the sense indicated above. Thus, the theoretical consistency of the evaluations of the maximum probability and the least squares not only is not accomplished in practice, but also leads to inconsistency of the evaluations of accuracy with possible deviations from the adopted distribution $F(\xi)$. /17

As is common knowledge, it often turns out in practice

that some percent of the measurements is burdened with large errors, which occur from the adopted distribution $F(\xi)$.

These measurements are usually called anomalous. The method of least squares, based on the assumption of the normalcy of the distribution $F(\xi)$, is especially sensitive to similar measurements.

In the capacity of an example, we will examine the solution of problem (5) using algorithm (7) of the method of least squares. We will assume that, among n measurements of \tilde{d}_i ($i=1, \dots, n$), there is some number m of anomalous measurements, which have the same error $\xi_i = A$ ($i=1, \dots, m$). The errors of the remaining $n-m$ measurements are distributed according to the law in (6). Then, the evaluation \hat{q} , determined according to formula (7), will have the additional error

$$\delta_{\text{add}}q = pA, \quad p = \frac{m}{n}, \quad (29)$$

associated with the presence of anomalous measurements.

It is evident from comparison of expressions (8) and (29) that, if there exists such an $\varepsilon > 0$ that the inequality $p > \varepsilon$ takes place, irrespective of the number n , then, with a sufficiently great n , the additional error (29) becomes decisive.

We will compare the obtained result with the solution of the problem according to the method of a minimum of moduli, with which the evaluation of \hat{q} is sought from the expression

$$\hat{q} = \arg \min_q \sum |\tilde{d}_i - q|. \quad (30)$$

As is common knowledge, here, the algorithm of evaluation is reduced to the search for the selective median of the set

of numbers \tilde{d}_i ($i=1, \dots, n$), i.e., the magnitudes of \hat{q} for which the number of measurements of \hat{d}_i , which satisfy the inequality $\tilde{d}_i \leq \hat{q}$, is equal to the number of measurements which satisfy the inequality $\tilde{d}_i \geq \hat{q}$ [5]. The additional error $\delta_{\text{add}} q$ of this algorithm, associated with the anomalous measurements indicated above, may be found from the equation

$$\int_0^{\hat{q}} \varphi(t) dt = \frac{m}{n-m} = \frac{p}{1-p} ,$$

where $\varphi(t) = \frac{\exp(-t^2/2)}{\sqrt{2\pi}}$ is the normed density of the normal distribution.

The approximated solution of this equation, with a sufficiently small p , gives

$$\delta_{\text{add}} q \approx p \sigma \sqrt{2\pi} . \quad (31)$$

It is evident from comparison of expressions (29) and (31) that the additional error of the method of least squares is proportional to the magnitude A of the anomalous error, whereas the additional error of the method of the minimum of moduli does not depend on this magnitude. Therefore, with $A \gg \sigma$, the second of these methods proves considerably less sensitive to anomalous errors of the examined type.

The strong dependence, shown in the simplest example, of the evaluations of the method of least squares on anomalous errors in measurements has a sufficiently general nature. It is associated with the principle of minimization of the quadratic form from the vector of discrepancies. Therefore, in practice, during evaluation according to the method of least squares, as a rule, one makes use of different empirical and semiempirical method of preliminary purification of the measurement information from coarse anomalous measurements. /19

/18

3. Contemporary Approach to the Problems of Evaluation

The aforementioned shortcomings in the classical methods of evaluation are determined by the fact that the actual distributions $F(\zeta)$, which are part of dependence (1) of the errors, differ from those used during construction and analysis of algorithms of evaluation. One of the methods of overcoming this circumstance consists of numerous attempts to make the distribution $F(\zeta)$ more precise, according to the results of measurements (so-called adaptive algorithms of evaluation). Although a number of interesting results is obtained in this way, its possibilities are limited. Even if we limit ourselves to finding the mathematical expectancy $E(\zeta)$ and the covariation matrix $D(\zeta)$, with simultaneous searching for the evaluation of $\hat{q} = \{q_1, \dots, q_m\}$, then it will be necessary to determine a total of $\frac{n(n+3)}{2} + m$ scalar magnitudes, according to $n+m$ measurements of the components of the vector \hat{d} . Apparently, this is impossible, since the number of determinable magnitudes considerably exceeds the number of measurable magnitudes. Therefore, all of the existing adaptive algorithms of evaluation are based on sufficiently strict assumptions on the form of the distribution $F(\zeta)$. The most widespread is its representation in form (3), with a fixed matrix L . In this case, the adaptation amounts to the search for the most suitable value of the factor σ^2 . It is evident that, in this posing of the problem, all of the aforementioned shortcomings of the method of least squares remain in force.

A principally new approach to the problem of evaluation consists of rejecting the representation of the distribution $F(\zeta)$ and replacing it with some set F , which includes the possible distributions of the vector ζ . We will use N to designate some scalar characteristic of the accuracy of evaluation of \hat{q} . For definiteness, we will consider that any in-

/20

crease in the accuracy of \hat{q} corresponds to any decrease in N . Then, in the capacity of a guaranteed characteristic N_{guar} of the accuracy (6), we will adopt its upper boundary in the set F , i.e., the magnitude which satisfies the inequality

$$N \leq N_{\text{guar}} \text{ with } F(\xi) \in \mathcal{F}. \quad (32)$$

If N_{guar} is an accurate upper limit, then we will speak of a strict guaranteed characteristic of accuracy. In this case

$$N_{\text{guar}} = \sup_{F(\xi) \in \mathcal{F}} N. \quad (33)$$

The search for a minimum of the strict guaranteed characteristic of accuracy serves as a criterion of optimality in problems of selecting a strategy of evaluation (i.e., the choice of an algorithm of evaluation, a plan of measurements and a mathematical model of the system being examined). Here, the optimal strategy S_{opt} is found using the dependence

$$S_{\text{opt}} = \arg \min_{S \in \Sigma} \left[\sup_{F(\xi) \in \mathcal{F}} N \right], \quad (34)$$

where the minimum is sought in the given set Σ of permissible strategies S .

We will note that the advisability of a similar approach to problems of evaluation was evidently clear even to classical scholars of mathematical statistics. However, the algorithms of evaluation based practically on this approach began to be created only in the middle of the 1960's. This is explained by the following reasons. /21

A. In recent years, requirements have sharply increased

for efficiency of evaluation and reliability of determination of the accuracy of obtained results. This is associated with the fact that evaluations of the parameters of physical phenomena are being used ever more widely during the solution of important applied problems (such as the control of movement of different systems, the determination of basic physical constants, etc.).

B. The switch to automatic methods of processing of measurement information on computers made it especially intolerable to utilize different empirical and semiempirical methods of information processing, based on the intuition of the researcher and requiring repeated intervention by man into the operation of computers.

C. The constant improvement of measurement and computer technology has stipulated the possibility of a sharp increase in the volume of measurement information utilized during evaluation. However, in this case, the effect of the indicated instability of the classical results is manifested especially strongly.

D. The utilization of modern computers and the algorithms of optimization developed in calculations for these machines has revealed the practical possibility of solving sufficiently complex problems of the type of (34) '

It is evident from dependences (33) and (34) that the solution of problems of evaluation, based on the approach indicated above, depends substantially on the type of set F of possible distributions of $F(\zeta)$, as well as the utilized characteristic of accuracy of W and the mathematical model (1). In this connection, there is not a general method for solving these problems at the present time. However, for individual

22

specific cases, sufficiently far-reaching results have been obtained, and effective numeric algorithms have been developed. A brief description of these results is given below.

4. Anti-Interference Evaluation

As was indicated above, one of the shortcomings of the method of least squares is the great effect of anomalous measurements on the accuracy of the obtained evaluations. The presence of this shortcoming, as well as the necessity of a new approach to problems of evaluation, were indicated in studies [7,8]. First proposed in study [9] was the method of construction of anti-interference evaluation, and its asymptotic optimality was shown in terms of (34), for one class F of possible distributions of $F(\xi)$, which are close (in a certain sense) to normal. This method was developed in a number of subsequent studies. In study [10], it was generalized to multiple regression problems. In this study, as well as in [11], they analyzed the dependence of the type of anti-interference algorithm on the set F of possible distributions of $F(\xi)$. One of the types of anti-interference algorithms, namely the method of the minimum of moduli (selective medians in a unidimensional case), was studied in detail in studies [5,12].

All of the indicated studies are based on the assumption that the vector d of the measurements is a set k of vectors (in the specific case—scalars) d_j ($j=1, \dots, k$), which are identically associated with q . The vectors of the errors ξ_j ($j=1, \dots, k$), which are assumed to be mutually independent and equally distributed, correspond to these vectors in dependence (1). Here, the magnitudes which are part of equality (1) may be represented in the form

$$\hat{d} = \begin{pmatrix} \hat{d}_1 \\ \vdots \\ \hat{d}_k \end{pmatrix}, d(q) = \begin{pmatrix} d(q_1) \\ \vdots \\ d(q_k) \end{pmatrix}, \xi = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix}, \quad (35)$$

/23

where the function $d'(q)$ determines the dependence of any of the vectors (scalars) d_j ($j=1, \dots, k$) on q .

In this case, the density $F(\zeta)$ of distribution of the vector ζ may be represented in the form

$$F(\zeta) = \prod_{j=1}^k F'(\zeta_j), \quad (36)$$

where $F'(\zeta_j)$ is the density of distribution of any of the vectors ζ_j ($j=1, \dots, k$).

In addition, it is assumed that the mathematical expectancy is

$$E(\zeta_j) = 0 \quad (j=1, \dots, k). \quad (37)$$

Here, the condition $F(\zeta) \in F$ is replaced by the condition

$$F(\zeta) \in F' \quad (j=1, \dots, k), \quad (38)$$

where F' is the set of possible densities of distributions of the vectors ζ_j ($j=1, \dots, k$).

The dispersion of the evaluation of any scalar function of the vector q is utilized as the characteristic N of the accuracy in dependences (32)-(34).

The essence of the examined methods of constructing anti-interference evaluations consists of the fact that within the set F' is selected the worst distribution of $F'(\zeta_j) \in F'$, in which the minimum of Fisher information is achieved. Constructed for this distribution is the corresponding algorithm of maximum probability, and it is shown that, asymptot-

24

ically (with $k \rightarrow \infty$), it satisfies the condition of optimality of the type of (34) [10,11].

Specifically, if the set of all distributions, for which the density $F'(0) > B > 0$, is viewed as F' , then Laplace's distribution is the corresponding worst distribution, and the method of the minimum of moduli is the optimal algorithm of filtration. In other words, this method corresponds to the case when practically nothing is known about the distribution $F'(\zeta_j)$.

Thus, the anti-interference properties of the method of the minimum of moduli, shown above in the simplest example, has a sufficiently deep theoretical basis. Also examined in studies [10,11] are other methods of representing the set F' , for which the corresponding optimal algorithms of evaluation are constructed.

On the whole, the examined method of constructing anti-interference evaluations is promising, since it reveals the possibility of automation of the process of exclusion of anomalous measurements, and, finally, makes it possible to reduce to a minimum the intervention of man in the solution of problems of evaluation on computers.

A shortcoming of this method is the fact that it is based on assumptions (35), (36), and (37). Here, the effect of correlation dependences between the vectors ζ_j ($j = 1, \dots, k$) is not taken into account at all, and neither is the possible spread of the mathematical expectancies of these vectors. As was indicated above, with a large number of utilized measurements, this effect on the evaluation of the accuracy \hat{q} becomes decisive. Demonstrated here is the inconsistency of evaluations of accuracy according to the adopted assumptions. The

25

latter fact is associated with the fact that the obtainable evaluations of \hat{q} , in the majority of cases, are theoretically consistent.

5. Evaluation with Restrictions According to an "Upturn" (Case of Random Correlation of Errors)

The aforementioned strong effect of the correlation dependences between the components of the vector ζ on the accuracy of the evaluation of \hat{q} , as well as the impossibility of the reliable determination of these dependences, were causes of the appearance of a considerable number of studies, in which the existence of a random worst correlation between the magnitudes of ζ_i ($i=1, \dots, n$) is assumed. In other words, it is assumed that the coefficients of correlation $K(\zeta_i, \zeta_i)$ may take on any values in the closed interval $[-1, 1]$. Here, different methods are utilized for representing the set F . The simplest of these are the so-called restrictions according to an "upturn", of the form

$$|\zeta_i| \leq \delta_i \quad (i=1, \dots, n), \quad (39)$$

where δ_i ($i=1, \dots, n$) are the given non-negative numbers.

Here, the guaranteed characteristics of accuracy (33) are determined by the expression

$$N_{\text{guar}} \max_{\mathcal{F}} |\delta \ell|, \quad (40)$$

where $\delta \ell$ is the error in evaluation of $\hat{\ell}$ of some scalar magnitude $\ell = \ell(q)$.

In the more general case, the following restrictions may be utilized in place of (39):

/26

$$\xi \in Q,$$

where Q is the given set in the space of the vectors ξ .

Often, in place of the conditions in (39), restrictions are set on the values of the dispersions of the magnitudes of ξ_i

$$\mathcal{D}(\xi_i) \leq \delta_i \quad (i=1, \dots, n). \quad (41)$$

Here, the possibility is assumed of the random correlation between different components of the vector ξ , and the following magnitude is utilized as the guaranteed characteristics of accuracy

$$N_{\text{guar}} \max_{\xi} \mathcal{D}(\xi_i). \quad (42)$$

One can show that, for a sufficiently broad class of algorithms of filtration, the search for an optimal strategy of evaluation, based on the minimization of expression (40) with the conditions in (39), is equivalent to the solution of the very same problem by means of minimization of expression (42) with the conditions in (41) [1].

Such an approach to problems of evaluation was first utilized in study [13]. Examined here was the problem of the selection of an optimal body of measurements, which provides the achievement of a minimum of expression (40) with the conditions in (39) and filtration according to the method of least squares. The problem was reduced to linear programming, for which there are well-developed algorithms of numerical solution. The basic properties of this solution were studied. This study served as an impetus for the conduct of a considerable number of studies in the indicated area. Exhaustively studied here

was the problem of the selection of an optimal linear unbiased algorithm of evaluation, and an optimal plan of measurements with conditions (39) and (40) or (41) and (42). /27

Developed here, along with a method for solving the problem of optimization, was a method which amounts to the finding of some generalized Tchebycheff polynomial which is deviates the least from zero. [14], as well as a method which utilizes the graphic geometric interpretation proposed in [15]. A listing of these studies and their results are located in [1,16].

The basic result of the indicated studies is proof of the fact that, with the assumptions made, only m of the appropriately selected measurements are optimally utilized (here, m is the dimensionality of the vector q). A further increase in the number of utilized measurements is not only of no use, but, generally speaking, leads to impairment of the guaranteed accuracy of the obtained evaluations. It is evident that, under these conditions, it is also impossible to speak of the consistency of the evaluation of \hat{q} . Thus, results were obtained which are fundamentally contradictory to the conclusions drawn on the basis of the classical assumption (3).

In those cases when the mathematical model (1) is constructed on the basis of the solution of the vector differential equation of movement

$$\frac{dq}{dt} = f(q, t) + \eta, \quad (43)$$

where t is the time and η is the interference vector, and conditions of the type of (39) or (41) may be imposed not only on ..

the vectors ξ of errors, but also on the vector η .

With similar assumptions, one concrete problem of evaluation of the accuracy of determination of movement was examined in [17]. In studies [18,19], this posing of the problem was utilized for the study of determination of the movement of a /28 spacecraft, using Kalman's filter. A detailed analysis of the problem of construction of an optimal recurrent algorithm of filtration, with a sufficiently general method of representation of the sets to which the vectors of errors of measurement and interference vectors may belong, is carried out in [20].

As an example, we will examine the problem of determining the velocity v of the change in some magnitude d at the initial moment of time $t=0$. Here, we will make use of the linear model (10), and assume that the errors ξ_i satisfy the condition

$$|\xi_i| \leq \Delta + \frac{w t^2}{2} \quad (i=1, \dots, n), \quad (44)$$

where $\Delta > 0$ and $w > \Delta$ are given numbers. The former of these characterizes the maximum possible error of the measurements, and the latter characterizes the maximum acceleration of the magnitude d , not taken into account by the linear model (17). This acceleration, generally speaking, is variable, and we are only given the maximum of its modulus.

We will find the evaluation of \hat{v} of the desired magnitude, using a linear algorithm of filtration:

$$\hat{v} = \sum_{i=1}^n x_i \tilde{d}_i, \quad (45)$$

where x_i ($i=1, \dots, n$) are, for now, the random coefficients of the algorithm, which satisfy the condition of empiricalness [1], which, in the examined case, may be written in the form

$$\sum_{i=1}^n x_i = 0, \quad \sum_{i=1}^n t_i x_i = 1. \quad (46)$$

Here, the error δv of the evaluation of \hat{v} is determined by the expression

$$\delta v = \sum_{i=1}^n x_i \xi_i; \quad (47)$$

Hence, making use of (44), we will find the guaranteed characteristics (40) of accuracy

1/29

$$N_{\text{guar}} = \max_S |\delta v| = \sum_{i=1}^n |x_i| \left(\Delta + \frac{wt_i}{2} \right). \quad (48)$$

In this posing of the problem, the selection of an optimal strategy S of evaluation in the set Σ of possible strategies is reduced to the search for the number n of measurements, as well as the times t_i and the coefficients x_i , which satisfy condition (46) and provide the achievement of a minimum of the magnitude in (48).

Switching to the solution, we will note that the dimensionality of the vector $q = \{a, v\}$ of state of model (17) $m=2$. Hence, making use of the results given above, we find that the minimum of expression (48) is achieved with a number $n=m=2$. Then, making use of (46), we will find the dependences between the times t_i of the measurements and the corresponding coefficients x_i ($i=1, 2$):

$$x_1 = -x_2 = -\frac{1}{t_2 - t_1}.$$

Substituting these expressions into (48), and seeking the minimum according to t_1 and t_2 , we obtain the solution of the problem.

$$n=2, t_1 = -t_2 = \sqrt{\frac{2\Delta}{w}}, x_1 = -x_2 = \frac{1}{2} \sqrt{\frac{w}{2\Delta}}. \quad (49)$$

Here, the following is achieved

$$\min_{\sum} N_{\text{guar}} = \min_{\sum} \max_{\mathcal{F}} |\delta \ell| = \sqrt{2\Delta w}. \quad (50)$$

We will note that, during the determination of the set Σ of possible strategies S , we do not impose any restrictions on the times t_i .

7 We will compare the obtained result with the evaluation according to the method of least squares, with the assumption of equal accuracy and non-correlation of the errors ξ_i . In this case, we will assume: that the measurements are carried out with a constant spacing τ , according to the time at the moments

$$t_i = i\tau, \quad (j=1, \dots, k), \quad (51)$$

where $k=n/2$ (n is the total number of measurements).

With these assumptions, the coefficients x_i ($i=1, \dots, n$) of algorithm (45) are determined by an equality of the type

$$x_i = \frac{t_i}{\sum t_i^2} = \frac{i}{2\tau \sum_{j=1}^k j^2}, \quad (52)$$

where, in the expression $\sum t^2$, the summation is carried out according to all times t_i .

Substituting (52) into (48), and making use of (50), as well as the known expressions for the sums of the orders of a natural series of numbers, we find

$$\bar{\gamma} = \frac{\bar{N}_{\text{guar}}}{\sum_{j=1}^k N_{\text{guar}}} = \frac{\beta \sum_{j=1}^k j + \frac{1}{\beta} \sum_{j=1}^k j^3}{2 \sum_{j=1}^k j^2} = \frac{3[2\beta + \kappa(\kappa+1)]}{4(2\kappa+1)\beta}, \quad (53)$$

where \bar{N}_{guar} is the value of the guaranteed characteristics (48)

of accuracy with evaluation according to the method of least squares and a uniform distribution of the times of the measurements, and

$$\beta = \sqrt{\frac{2\Delta}{\omega\tau^2}} . \quad (54)$$

We will record the number k and search for the magnitude of β which ensures achievement of $\min_{\beta} \gamma$. It is easy to show that this will be the case with

$$\beta = \sqrt{\frac{k(k+1)}{2}} \quad (55) \quad /31$$

Substituting this value of β into (53), we find

$$\min_{\beta} \gamma(k) = \frac{3\sqrt{k(k+1)}}{(2k+1)\sqrt{2}} \quad (56)$$

It is easy to assert that, with an increase in k , $\min_{\beta} \gamma(k)$ increases monotonously from a magnitude $\min_{\beta} \gamma(1) = 1$ to $\min_{\beta} \gamma(\infty) =$

$\frac{3}{2\sqrt{2}} \approx 1.06$. The lower boundary of this interval corresponds to the total coincidence of the examined algorithm of evaluation, according to the method of least squares, with the optimal linear unbiased algorithm given above. We will note that, with a sufficiently large k , equality (55) may be replaced with the approximate dependence

$$T \approx \sqrt{2} T_0, \quad (57)$$

where $T = 2\pi k$ and $T_0 = t_2 - t_1$, are the time intervals within which the measurements are carried out, with evaluation according to the method of least squares and with utilization of the optimal algorithm (49), respectively.

Based on the obtained results, one can draw the following conclusions.

A. With planning of the experiment in accordance with dependence (55) or (57), evaluation according to the method of least squares, with uniform distribution of the times t_i with any number of measurements, is practically equivalent to the optimal algorithm (49). This circumstance may be utilized in that case when, along with the errors ζ_i , which satisfy the conditions in (44), mutually non-correlated errors ζ'_i are part of the righthand portions of the dependences in (17)!
Then, making use of the method of least squares, with a sufficiently large number n of utilized measurements, one can practically preclude the effect of the errors ζ'_i . /32

B. With a deviation from the optimal relationships (55) or (57) in one direction or the other, the guaranteed accuracy of evaluation, according to the method of least squares, is impaired unrestrictedly.

C. The best guaranteed accuracy of the magnitude is determined by expression (50), which does not depend on the plan of the experiment. Thus, with the examined posing of the problem, consistent evaluations of \hat{v} do not exist.

The conclusions obtained here on the simplest example have a sufficiently general nature. Experience in the use of evaluation "according to an upturn" applied to different, sufficiently complex problems shows that the method of least squares is practically equivalent to the optimal linear unbiased algorithm of evaluation in that case when the planning of the experiment is accomplished on the basis of the methods set forth above. With non-observance of this conditions, the guaranteed accuracy of evaluation, according to the method of least squares, may be considerably worse than the optimal accuracy.

This conclusion follows from the fact that any linear

unbiased evaluation is equivalent to the solution of problem (1) according to the method of least squares, using the expression.

$$\hat{q} = \arg \min_q [\tilde{d} - d(q)]^T P [\tilde{d} - d(q)], \quad (58)$$

where P is the weighted matrix, selected by the appropriate method [26]. The difference between the classical algorithm (4) and expression (58) consists of the fact that, in the former case, the weighted matrix is determined by the equality $P=L^{-1}$, which follows directly from the well-known Gauss-Markov theorem [1], and in the latter case, the weighted matrix can be found from the solution of the corresponding problem of linear programming. /33

Thus, for the solution of applied problems, one can successfully make use of the well-developed algorithms of the method of least squares (with the condition of preliminary processing of anomalous measurements). However, planning of the experiment and evaluation of the accuracy of the obtained results should be carried out with regard for the potential spreads of the distributions of the probabilities of the errors ζ .

6. Evaluation with Restrictions on the Elements of the Mathematical Expectancy and the Covariational Matrix of Errors

Based on the methods described in the preceding section, one can obtain sufficiently reliable guaranteed evaluations of the accuracy of determination of the vector \hat{q} , according to the results of measurements, as well as solving the problem of the selection of an optimal linear unbiased algorithm of filtration and an optimal plan of measurements, which ensure a maximum guaranteed accuracy of \hat{q} . Therefore, these methods are presently beginning to be widely utilized during

the solution of vital problems, which place high requirements on the accuracy and reliability of the evaluation (for example—for determining the initial data for controlling the flight of a spacecraft). A shortcoming of these methods is the fact that they often give excessively less sensitive evaluations of the accuracy of \hat{q} . In this connection, methods have begun to be developed in recent years for solving the examined problems, assuming that the coefficients $K(\xi_i, \xi_j)$ of correlation between the components of the vector ξ are not random, /34 but lie within some boundaries.

In the simplest variant of the indicated posing of the problem, the problem of construction of the optimal linear unbiased algorithm of evaluation, and selection of the optimal plan of the measurements was solved in [21]. The results of this study are set forth in detail in [1]. Here, it is assumed that the set F of possible distributions of $F(\xi)$ is determined by the inequalities in (41) and the condition

$$|K(\xi_i, \xi_j)| \leq k, \quad i \neq j, \quad (59)$$

where $0 \leq k \leq 1$ is a given number, and expression (42) is utilized as the guaranteed characteristics of accuracy.

Through these assumptions, it was proved that the optimal situation is the utilization of the method of least squares, with the condition of concentration of all of the points measured in m (m is the dimensionality of the vector q), determined by the solution of some problem of linear programming. With an increase in the total number n of measurements, the accuracy of evaluation increases; however, with $k > 0$

$$\lim_{n \rightarrow \infty} \max_{\mathcal{F}} D(\delta \ell) > 0,$$

i.e., the evaluation of \hat{q} is not consistent. In this case, beginning with some value of n , a subsequent increase in the number of measurements leads to such an insignificant increase in accuracy that it can be considered practically useless.

The indicated trend received further development in study [22], in which condition (59) is replaced by a restriction of the type

$$k_1 \leq K(\xi_i, \xi_j) \leq k_2 ,$$

where k_1 and k_2 are given numbers.

The problem of a guaranteed evaluation was examined in a considerably more general posing in studies [23, 24, 25], in which the following restrictions, which determine the set of possible distributions, are placed on the elements of the mathematical expectancy $E = \{E(\xi_i), \dots, E(\xi_j)\}$ and the covariation matrix $D(\xi) = (D_{ij})$ of the errors.

$$| E(\xi_i) | \leq M_i , \quad (60)$$

$$| D_{ij} - v_{ij} | \leq \theta_{ij} \leq D_{ij} + v_{ij} , \quad (61)$$

135

where M_i , D_{ij}^* , and $v_{ij} \geq 0$ ($i, j = 1, \dots, n$) are given numbers.

In place of condition (61), one can examine the following restrictions, placed on the dispersion $D(\xi_i)$ and the coefficients of correlation $K(\xi_i, \xi_j)$

$$D(\xi_i) \leq \delta , \quad K_{ij} - w_{ij} \leq K(\xi_i, \xi_j) \leq K_{ij}^* + w_{ij} , \quad (62)$$

where δ_i , K_{ij}^* , and $w_{ij} \geq 0$ ($i \neq j, i, j = 1, \dots, n$) are given numbers.

The following magnitudes are utilized as guaranteed characteristics of accuracy

$$m_{vp} = \max_{\mathcal{F}} |E(\delta l)|, \quad \mathfrak{D}_{gu} = \max_{\mathcal{F}} \mathfrak{D}(\delta l). \quad (63)$$

Two variants of the criteria of optimality are examined, utilized during the selection of a strategy S of the solution of the examined problem in the set Σ of possible strategies.

In the first variant, we are searching for

$$\min_{\Sigma} \beta_{gu}^2 \quad \beta_{gu}^2 = \max_{\mathcal{F}} E(\delta^2 l) = m_{gu}^2 + \mathfrak{D}_{gu}. \quad (64)$$

In the second variant, the following is found /36

$$\max_{\Sigma} H_{guar} \quad H_{guar} = \min_{\mathcal{F}} P(|\delta l| \leq \alpha). \quad (65)$$

Here, β_{guar}^2 is the guaranteed value of the mathematical expectancy of the square of the error δl , H_{guar} is the guaranteed reliability of the realization of the inequality $|\delta l| \leq \alpha$, α is the permissible maximum error of evaluation of \hat{l} , and $P(\cdot)$ is the probability of realization of some event.

The first of these variants does not require any sort of assumptions on the distribution of the errors δl . In order to utilize the second variant, it is necessary to have this distribution as a given. If it is normal, then one can write that, with $m_{guar} \leq \alpha$ [1]

$$H_{guar} = \Phi\left(\frac{\alpha + m_{guar}}{\sqrt{\mathfrak{D}_{guar}}}\right) + \Phi\left(\frac{\alpha - m_{guar}}{\sqrt{\mathfrak{D}_{guar}}}\right), \quad (66)$$

where

$$\Phi(x) = \frac{1}{2\pi} \int_0^x e^{-\frac{t^2}{2}} dt.$$

It is easy to assert that the posings of the problem used

in paragraphs 2 and 5 of the present study are particular cases of the general posing given here. Actually, assuming that, in expressions (60) and (61),

$$M_i = 0, v_{ij} = 0, (D_{ij}^*) = \sigma^2 L \quad (i, j = 1, \dots, n)$$

and taking into account the fact that, here, $m_{\text{guar}} = 0$ and $\beta^2 = D_{\text{guar}} = D(\delta_l)$, we obtain the classical posing of the problem, which underlies the method of least squares.

On the other hand, assuming that

$$D_{ij}^* = 0, v_{ij} = 0 \quad (i, j = 1, \dots, n)$$

and making use of the magnitude $m_{\text{guar}} = \max_F |\delta_l|$ as the characteristics of accuracy, we obtain a posing of the problem which leads to evaluation according to an "upturn".

137

Thus, the posing of the problem given here makes it possible to construct a broad spectrum of solutions of the problems of evaluation, which include the method of least squares and evaluation according to an "upturn" as specific cases. The selection of one variant or another within this spectrum depends on the existing information on the mathematical expectancy and the covariation matrix of the errors ζ , as well as the required reliability of the solution of the problem of evaluation.

With the aforementioned assumptions (60) and (61) or (60) and (62), the following basic results were obtained in studies [23, 24, 25].

A. For a random linear (or linearized) unbiased algorithm of evaluation, the error δ_l of which can be represented in the form [1]

$$\delta_l = X \xi, \quad (67)$$

where $X = (X_1, \dots, X_n)$ is some matrix with a $1 \times n$ dimension, expressions are written down for the guaranteed characteristics of accuracy m_{guar} and D_{guar} which are the upper boundaries of the magnitudes of $|E(\delta \ell)|$ and $D(\delta \ell)$ in the set F . The first of these characteristics is strict in terms of (33), and satisfies the first of the equations in (63). As far as the second characteristic is concerned, we determine the conditions of strictness which are necessary and sufficient, and show that, in this case, it satisfies the second of the equalities in (63).

B. For the case when the mathematical model (1) is linear relative to q , we examine the problem of construction of a linear unbiased algorithm of filtration and selection of a plan of measurements which are optimal according to one of the criteria of (64) or (65). It is shown that it comes down to the problem of quadratic (in specific cases—linear) programming, for which there is a numerical algorithm of solution. Some properties of this solution are examined. Specifically, it is shown that, in the general case, the obtained evaluation is inconsistent. The conditions are determined under which the optimal plan of measurements is reduced to the accumulation of measurements at some finite number of points. /38

The obtained results are naturally extended to the case when the mathematical model (1) is constructed based on the solution of the vector differential equation (43).

As an example, we will examine the determination of the velocity v , based on model (17), assuming that the errors with restricted values of the dispersions and the coefficients of correlation are added to errors which satisfy the conditions in (44). Here, the set F of possible distri-

butions of the errors ζ is determined by the following conditions, imposed on the mathematical expectancies $E(\zeta_i)$, the dispersions $D(\zeta_i)$, and the coefficients of correlation $K(\zeta_i, \zeta_j)$ of the components ζ_i of the vector ζ ,

$$|E(\zeta_i)| \leq \Delta + \frac{wt^2}{2}, \quad D(\zeta_i) \leq \sigma^2, \quad |K(\zeta_i, \zeta_j)| \leq K \quad (i \neq j; i, j = 1, \dots, n), \quad (68)$$

where $\Delta \geq 0$, $w \geq 0$, $\sigma^2 \geq 0$, and $1 \geq K \geq 0$ are given numbers.

Given in study [24] is the solution of the problem of selecting a linear unbiased algorithm for constructing the evaluation of \hat{v} with the assumptions in (68), and also assuming that the number n of utilized measurements is given (or restricted at the top). In this case, optimization is carried out according to criterion (64). As a result, it is shown that the optimal plan envisages the conduct of all measurements at two moments in time t_1 and t_2 . Here,

$$\left. \begin{array}{l} t_1 = -t_2 = -\frac{T_0}{2}, \quad T_0 = 2 \sqrt{4 \frac{\sigma^2 + \Delta^2}{w t^2}}, \\ n_1 = n_2 = \frac{n}{2}, \quad \bar{x}_1 = -\bar{x}_2 = -\frac{1}{T_0}, \end{array} \right\} \quad (69)$$

139

where T_0 is the total duration of the interval of measurements, n_i are the number of measurements at the moments t_i , \bar{x}_i are the sums of the coefficients x_i of algorithm (45) of evaluation ($i=1, 2$), which correspond to these moments, and

$$\gamma = \sigma \sqrt{\frac{1-K}{n} + K}. \quad (70)$$

Here

$$\min \sum \beta_{\text{guar}}^2 = w \left(\sqrt{\gamma^2 + \Delta^2} + \Delta \right). \quad (71)$$

We will compare the obtained result with the evaluation according to the method of least squares, with uniform distribution of the times of the measurements in the closed interval

$$-\frac{T}{2} \leq t_i \leq \frac{T}{2}, \quad (72)$$

where T is the given duration of the interval (72).

Given in [24] is the asymptotic expression, corresponding to this case, for β_{guar}^2 , which is correct with a sufficiently large n

$$\beta_{\text{guar}}^2 \approx 9 \left(\frac{\sigma^2 + \Delta^2}{T^2} + \frac{w^2 T^2}{256} + \frac{w \Delta}{8} \right). \quad (73)$$

The minimum of this expression is achieved with the condition (57). Here, $\beta_{\text{guar}} \approx 1.06 \min_{\epsilon} \beta_{\text{guar}}$, where the magnitude of $\min_{\epsilon} \beta_{\text{guar}}$ is determined by expression (71).

Thus, the switch from the optimal algorithm of filtration to the method of least squares, with uniform distribution of the times of the measurements, impairs the magnitude of β_{guar} only slightly. However, this result is achieved only with the correct selection of the duration of the T -dimensional interval. It is evident from (73) that the deviation from condition (57) in one direction or the other may lead to a sharp impairment of the guaranteed accuracy of evaluation of \hat{v} . /40

Given in [25] is the solution of this very same problem with replacement of condition (64) of optimality by condition (65). Here, the law of distribution of errors δ_t is assumed to be normal. The optimal plan, obtained as a result, also provides for the conduct of all measurements at two moments in time t_1 and t_2 . Here,

$$t_1 = -\frac{T_1}{2}, \quad h_1 = h_2 = \frac{n}{2}, \quad \bar{x}_1 = -\bar{x}_2 = -\frac{1}{T_1},$$

where T_1 is the optimal duration of the interval of measurements, for the determination of which a numerical algorithm

is given. Also compared are the values of the maximum errors α , at which an identical guaranteed reliability H_{guar} determined by expression (65), is provided. It is shown that, in the examined problem, with $H_{\text{guar}} \leq 0.999$, the switch from the criterion in (65) to the criterion in (64) entails an increase in the maximum error α of no more than 17%.

Thus, with the assumption of the normalcy of distribution of the evaluation of $\delta\ell$, optimization according to the criterion in (65) makes it possible to somewhat improve the preciseness of the value of the maximum error α with a given guaranteed reliability (by comparison with the utilization of the criterion in (64)).

With random distribution of the errors $\delta\ell$, the magnitude of β_{guar} , determined by expression (64), sufficiently completely characterizes the accuracy of evaluation of $\hat{\ell}$. Specifically, in this case, it is associated with the guaranteed reliability H_{guar} and the maximum error by the following relationship, which follows directly from Tchebycheff's inequality [2] /41

$$\sqrt{H_{\text{guar}}} \leq 1 - \left(\frac{\alpha}{\beta_{\text{guar}}}\right)^2. \quad (74)$$

Therefore, in the examined example, expression (71) characterizes the maximum accuracy of evaluation of \hat{v} , with a given number n of measurements. Hence, making use of (70) and assuming that $n \rightarrow \infty$, we find the magnitude

$$\chi = \inf_n \min_{\sum \beta_{\text{guar}}} \beta_{\text{guar}} = \sqrt{w(\sqrt{w\sigma^2 + \Delta^2} + \Delta)}, \quad (75)$$

which determines the precise lower boundary of β_{guar} in the set of all linear unbiased algorithms of evaluation with a random number n of measurements. This magnitude determines the best information content of the evaluation of \hat{v} , inherent

in the given problem. This magnitude depends on the parameters w , k , σ , and Δ , which are part of condition (68) and determine the set F of possible distributions of the errors ζ .

It is evident that the equality

$$\chi = 0$$

(76)

is a necessary and sufficient condition of the possibility of constructing a consistent linear unbiased algorithm of evaluation. This equality may occur with one of the following conditions:

A. $w=0$, $k\sigma \neq 0$, $\Delta \neq 0$. From dependences (69) and (70), it is evident that, in this case, consistency is achieved because of the unrestricted increase in the duration T of the /42 interval of measurements.

B. $w \neq 0$, $k\sigma=0$, $\Delta=0$. Consistency is achieved because of the fact that $T_0 \rightarrow 0$.

C. $w=0$, $k\sigma=0$, $\Delta=0$. Consistency is achieved with a random T_0 only because of the fact that $n \rightarrow \infty$. We will note that a specific case of this condition is the simultaneous fulfillment of equalities $w=k=\Delta=0$. If, in this case, one replaces inequality $D(\zeta_i) \leq \sigma^2$ with equality $D(\zeta_i) = \sigma^2$, then the dependences in (68) are transformed into assumptions which lead to an algorithm of evaluation according to the method of least squares.

Thus, in the examined problem, a consistent evaluation can only be obtained in some idealized particular cases, which almost never take place in practice.

With $k\sigma=0$, the righthand portion of equality (75) coin-

cides with the corresponding expression (50), obtained on the basis of restriction according to an "upturn".

We will examine also the case $\Delta=0$. Here, expression (75) takes on the form

$$X = w^{\frac{1}{2}} \zeta^{\frac{1}{2}} k^{\frac{1}{4}}.$$

Thus, with $\Delta=0$ and assuming the random correlation between the components of the vector of the errors, $\zeta \sim w^{\frac{1}{2}} \zeta^{\frac{1}{2}}$. With restriction of the moduli of the coefficients of correlation of the magnitude, the value of X diminishes in proportion to $k^{\frac{1}{4}}$.

It is evident from the examined example that the guaranteed accuracy of evaluation depends substantially on the restrictions which determine the set F of possible distributions of $F(\zeta)$. Here, the placement of more rigid restrictions makes it possible to improve the guaranteed evaluation of accuracy.

43

7. Statistical Simulation of Problems of Evaluation

The above-described methods for solving problems of evaluation have strict mathematical bases, which are based on some assumptions about the ideal mathematical model $d=d(q)$, the set F of possible errors ζ , and the set Σ of permissible strategies of the solution of the problem. In the general case, when these assumptions are not fulfilled, one can use the method of statistical simulation, which consists of the following.

We will examine problem (1) with some given $d=d(q)$, F , and Σ . We will be given the algorithm of $\hat{q}=\hat{q}(d)$ and the composition \tilde{d} of the measurements which belong to the set of permissible strategies. They can be obtained from the solution

of the strict problem, with some idealized assumptions, or found on the basis of heuristic considerations. In addition, we will be given the nominal value q_0 of the vector q . Making use of the generator of random numbers, and taking into account the condition $F(\zeta) \in \mathcal{F}$, we will construct a random set of vectors ζ_j ($j=1, \dots, N$) of the errors, and the corresponding model set of measured vectors $\tilde{d}_j = d(q) + \zeta_j$ ($j=1, \dots, N$). Then, we will compute the model evaluations of $\hat{q}_j = \hat{q}(\tilde{d}_j)$ and find their errors $\delta q_j = \hat{q}_j - q_0$ ($j=1, \dots, N$). The statistical characteristics of the set of these errors are utilized for the evaluation of the quality of the examined solution of the problem of evaluation. By carrying out similar computations for different compositions of measurements and algorithms of evaluation, one can select the optimal ones among them. Here, optimization can be carried out not only according to the criterion of achieving maximum accuracy of evaluation of \hat{q} , but also from other considerations (simplicity of the utilized algorithms, their rapid action, etc.). /44

An advantage of this method is the possibility of its utilization for the most varied assumptions and criteria of optimality. A shortcoming is the impossibility of obtaining mathematically strict solutions. Here, only the more or less "plausible" results can be found (in the sense that it is understood in [27]). In spite of this, the method of statistical simulation is utilized successfully for the solution of applied problems of evaluation [28, 29, 30].

8. Conclusion

It follows from the results given above that notable successes have been achieved in recent years in the development of methods of evaluation, based on the assumption that the distributions of the errors in the initial data are not precisely known, and belong to some given sets. Here, the

basic progress has taken place in the following two areas.

A. It is assumed that the vector of measurements d is a set of a finite number of vectors d_j , which are identically associated with the vector q of state. The errors which correspond to these vectors are assumed to be identically distributed and mutually independent. Also given is a set F' of possible distributions $F'(\xi_j)$ of these errors, and we seek an algorithm of evaluation, which possesses asymptotic effectiveness with the "worst" distribution $F'(\xi_j)$ in the set F' . In this case, "worst" is understood to mean such a distribution $F'(\xi_j) \in F'$ with which a minimum of Fisher information is achieved. It is shown that algorithms constructed in this way possess good stability with respect to interference, caused by anomalous errors ξ_i . Methods are developed for constructing similar anti-interference algorithms with different methods of representing the set F' . The number of such algorithms includes the method of least moduli, which has been developed in sufficient detail. /45

A shortcoming of a similar approach is the fact that possible dependences between different vectors of ξ_j are not taken into account in it, and neither are possible differences between the distribution of these vectors and the deviation of their mathematical expectancies from the adopted a priori values. This leads to consistency of the obtained evaluations. From the theoretical point of view, the consistency of the evaluation can be shown to be attractive, but in practice, with the utilization of a sufficiently large number of measurements, it leads to instability of the evaluations of the accuracy of the obtained results. This, in turn, may be a cause of obtaining unjustified and optimistic evaluations of accuracy, as well as coarse errors during the solution of problems of experiment planning.

B. The set F of possible distributions $F(\xi)$ is deter-

mined by some restrictions, placed on the values of the elements of the vector of the mathematical expectancy $E(\zeta)$ and the covariation matrix $D(\zeta)$ of the errors .

A particular case of such an assumption is the restriction of the area to which the errors ζ may belong in the corresponding vector space. With these assumptions, one can determine the guaranteed accuracy of the evaluation of \hat{q} , obtained using any linear (or linearized) unbiased algorithm of filtration. Also solved is the problem of selecting an optimal linear unbiased algorithm of filtration and constructing the corresponding optimal plan of the experiment. Here, utilized in the capacity of a criterion of optimality, is either the achievement of a minimum of the guaranteed mathematical expectancy of the square of the error $\delta\hat{q}$ in the evaluation of \hat{q} of some scalar parameter $q = q(\hat{q})$, or a maximum of the guaranteed reliability of the fact that $\delta\hat{q} \leq \mathcal{E}$, where \mathcal{E} is the given maximally-permissible error in the evaluation of \hat{q} . This problem comes down to the well-developed algorithm of quadratic (linear) programming. Its solution is equivalent to the search for an optimal weighted matrix during the evaluation according to the method of least squares. Such a method is successfully used to obtain reliable guaranteed evaluations of the accuracy of the magnitude of \hat{q} with given possible spreads of the mathematical expectancy and the covariation matrix of the errors in the initial data, as well as to construct the corresponding optimal algorithms of evaluation and plans of the measurements.

As shortcoming of such an approach is the fact that the problem is solved in a set of all possible linear unbiased algorithms of evaluation. This is equivalent to the assumption of normalcy of the distribution of the errors ζ . As was indicated above, this is the cause of the strict dependence of the accuracy of the obtained evaluations of interference

evoked by anomalous measurements. In addition, it should be kept in mind that the solution of the above-indicated problems of optimization depends on the selection of the parameter $\ell = \ell(q)$, the accuracy of which we are interested in. Therefore, if the necessity occurs of optimizing the accuracy of the evaluations of several parameters $\ell_j = \ell_j(q)$ ($j=1, \dots, k$), then, in the general case, it is necessary to construct several algorithms $\hat{q}_j = \hat{q}_j(d)$, each of which optimizes the guaranteed accuracy of the corresponding evaluation. We will note that, in a number of specific cases, it is possible to successfully find the universal optimal strategy of solution of the problem, which only slightly impairs the accuracy of the evaluations of \hat{q}_j ($j=1, \dots, k$), as compared with their best achievable accuracy. 47

Unfortunately, not a single study is known to us which would unite both of the above-indicated approaches and solve the problem of optimal guaranteed evaluation, with simultaneous regard for the possible spreads of the types of distribution of $F(\zeta)$, the possible dependences between the distributions of the probabilities of different components of the vector ζ , and the spreads of their mathematical expectancies. The solution of the examined problem, with such general assumptions, is obviously a matter of the future. While such a general solution has not been achieved, one can recommend the following sequence of anti-interference guaranteed evaluation.

A. Found in the first stage is the preliminary evaluation of \hat{q}_{pr} , for which the algorithm $\hat{q} = \hat{q}(\tilde{d})$ is utilized, which possesses sufficient interference-resistance. As a function of the nature of the possible interference, this may be either the usual method of least squares, or one of the non-linear anti-interference algorithms of evaluation (for example, the method of the minimum of the moduli). Processing of the

available anomalous measurements is carried out based on the obtained result.

B. The mathematical model (1) of the problem is linearized relative to the found approximate value of $q_0 = \hat{q}_{pr}$, and one or several optimal linear unbiased algorithms $\hat{q}_j = \hat{q}_j(\tilde{d})$ are constructed, which provide the best guaranteed accuracy of the evaluations of the given parameters $\ell_j = \ell_j(q)$ ($j=1, \dots, k$). 48

A similar method for solving the problem of evaluation is considerably more time-consuming than the normal method of least squares. In addition, it requires detailed preliminary analysis of the possible errors ξ for the correct selection of the set F of possible distributions of these errors. However, for the solution of sufficiently vital problems and the utilization of modern computer technology, it may be considered fully justified.

REFERENCES

1. El'yasberg, P. Ye., Opredelenie dvizheniya po rezul'tatam izmereniy [Determination of Movement According to the Results of Measurements], Moscow, "Nauka", 1976.
2. Rao, S. R., Lineynye statisticheskie metody i ikh primenenie [Linear Statistical Methods and Their Application], Moscow, "Nauka", 1968.
3. Le-Cam, L., "On Some Asymptotic Properties of Maximum Likelihood Estimates and Relates Bayes' Estimates", Univ. Calif. Public. Statist. 1(11), 277-330 (1953).
4. Gurin, L. S., "O sostoyatel'nosti otsenok metodamnaimen'-shikh kvadratov" [The Consistency of Evaluations of the Method of Least Squares], in the collection: Matematicheskoe obespechenie kosmicheskikh eksperimentov [Mathematical Provision of Space Experiments], Moscow, "Nauka", 1978.
5. Mudrov, V. I., Kushko, V. L., Metody obrabotki izmereniy [Methods for Processing Measurements], Moscow, "Sovetskoe Radio", 1976.
6. El'yasberg, P. Ye., "Garantirovannaya otsenka tochnosti opredeleniya dvizheniya kosmicheskikh apparatov" [Guaranteed Evaluation of the Accuracy of Determination of the Movement of Spacecraft], Kosmicheskie issledovaniya 12(1), (1974).
7. Tukey, J. W., "A Survey of Sampling from Contaminated Distributions", In: Contributions to Probability and Statistics, Ed. by I. Olkin, Acad. Press, 1960.
8. Tukey, J. W., "The Future of Data Analysis", Ann. Math. Stat. 33(1), 331 (1962).
9. Huber, P. J., "Robust Estimation of a Location Parameter", Ann. Math. Stat. 35(1), 73 (1964).
10. Polyak, B. T., Tsypkin, Ya. Z., "Pomekhoustoychivaya identifikatsiya" [Anti-Interference Identification], in the Collection: Identifikatsiya i otsenka parametrov sistem [Identification and Evaluation of the Parameters of a System], I, 190, Preprints IV of the Symposium of IFAK, Tbilisi, "Mitsniereba", 1976.
11. Tsypkin, Ya. Z., Polyak, B. T., "Ogrublennyj metod maksimal'nogo pravdopodobiya" [Rough Method of Maximum Probability], Inter-Institution Collection: Dinamika sistem [System Dynamics], 12, Moscow, 1977.

12. Gil'bo, Ye. P., Chelpanov, I. B., Obrabotka signalov na osnove uporyadochennogo vybora. [Processing of Signals, Based on Ordered Selection], Moscow, "Sovetskoe radio", 1975.
13. Lidov, M. L., "K apriornym otsenkam tochnosti opredeleniya parametrov po metodu naimen'shikh kvadratov" [A Priori Evaluations of the Accuracy of Determination of Parameters, According to the Method of Least Squares], Kosmicheskie issledovaniya 2(5) (1964).
14. Belousov, L. Yu., Krupen', V. Ya., "O nekotorykh asimptoticheskikh otsenkakh nachal'nykh parametrov pri izmerenii dal'nosti" [Some Asymptotic Evaluations of the Initial Parameters, with Measurement of Range], Kosmicheskie issledovaniya 8(5) (1970).
15. Elfing, G., "Optimum Allocation in Linear Regression Theory", Ann. Math. Statist., 23, 255 (1952).
16. Bazhinov, I. K., Pochukaev, V. N., Optimal'noe planirovaniye navigatsionnykh izmereniy v kosmicheskem polete [Optimal Planning of Navigational Measurements in Space Flight], Moscow, "Mashinostroenie", 1976.
17. Novak, B. L., "O vliyanii momentov sil na tochnost' opredeleniya orientatsii iskusstvennykh sputnikov" [The Effect of Moments of Force on the Accuracy of Determination of the Orientation of Artificial Satellites], Kosmicheskie issledovaniya 10(2) (1972).
18. Hutchinson, C. E., D'Appolito, J. A., Bongiovani, P. L., "Minimax Design of Kalman-like Filters in the Presence of Large Parameter Uncertainties", AIAA Paper, 1972, No. 878.
19. Nishimura, I., "Worst-Error Analysis of Batch Filter and Sequential Filter in Navigation Problems", Journal of Spacecraft and Rockets 12(3), 133 (1975).
20. Kurzhanskiy, A. B., Upravlenie i nablyudenie v usloviyakh neopredelennosti [Control and Observation Under Conditions of Ambiguity], Moscow, "Nauka", 1977.
21. Bazhinov, I. K., Pochukaev, V. N., "Geometricheskaya interpretatsiya zadach otsenki tochnosti opredeleniya traektoriy kosmicheskikh apparatov" [Geometric Interpretation of Problems of Evaluation of the Accuracy of Determination of the Trajectory of Spacecraft], Kosmicheskoe issledovaniya 9(2) (1971).
22. Bakhshyan, B. Ts., "Nekotorye zadachi otsenki tochnosti prognozirovaniya parametrov traektorii i algoritmy ikh

resheniya" [Some Problems of Evaluation of the Accuracy of Forecasting of the Parameters of the Trajectory and the Algorithms for their Solution], Kosmicheskie issledovaniya 12(2) (1974).

23. Bakhshiyan, B. Ts., Nazirov, R. R., El'yasberg, P. Ye., "O garantirovannykh kharakteristikakh tochnosti opredeleniya parametrov dvizheniya kosmicheskikh apparatov" [Guaranteed Characteristics of Accuracy of the Determination of Parameters of Movement of Spacecraft], Kosmicheskie issledovaniya 15(4), 546 (1977).
24. Bakhshiyan, B. Ts., Nazirov, R. R., El'yasberg, P. Ye., Optimizatsiya opredeleniya orbity pri nepolnom znanii kovariatsionnoi matritsy i matematicheskogo ozhidaniya oshibok [Optimization of the Determination of the Orbit with Incomplete Knowledge of the Covariation Matrix and Mathematical Expectancy of the Errors], Vol. 15, No. 5, 658 (1977).
25. Bakhshiyan, B. Ts., Nazirov, R. R., El'yasberg, P. Ye., Optimizatsiya garantirovannoy nadezhnosti opredeleniya parametrov orbity v usloviyakh neopredelennosti [Optimization of the Guaranteed Reliability of the Determination of Parameters of an Orbit Under Conditions of Ambiguity], Vol. 15, No. 6, 835 (1977).
26. Rao, C. R., Unified Theory of Linear Estimation, Sankhya, Series A, No. 33, 371 (1971).
27. Poya, D., Matematika i pravdopodobnye rassuzhdeniya [Mathematics and Probability Discussions], Moscow, "Nauka", 1975.
28. Gurin, L. S., Tsoy, K. A., "Primenenie statisticheskogo modelirovaniya pri razrabotke i srovnitel'nom analize metodov otsenki parametrov" [Use of Statistical Simulation During the Development and Comparative Analysis of Methods of Parameter Evaluation], Collection: Obrabotka kosmicheskoy informatsii [Processing of Space Information], Moscow, "Nauka", 1976.
29. Burgin, M. S., Gurin, L. S., Tsoy, K. A., "Otsenki parametrov modifitsirovannogo raspredeleniya Puassona" [Evaluations of the Parameters of a Modified Poisson Distribution], Collection: Matematicheskoe obespechenie kosmicheskikh eksperimentov [Mathematical Provision of Space Experiments], Moscow, "Nauka", 1978.
30. Gurin, L. S., Tsoy, K. A., Ustoychivye (robastnye) uslovno effektivnye otsenki parametrov [Stable, Conditionally Effective Evaluations of Parameters], IKI AN SSSR Preprint, 382, 1977.

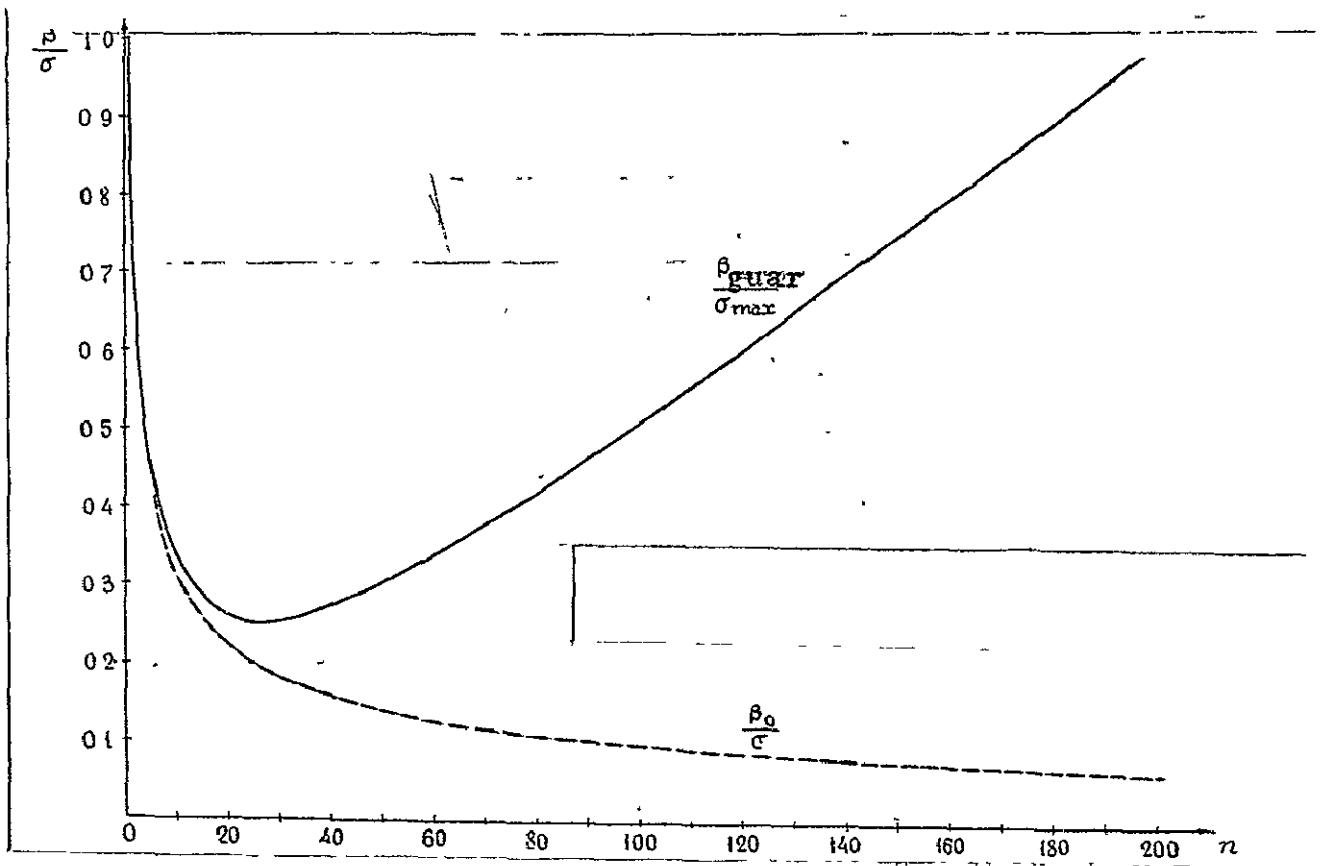


Figure 1
 Dependence of the Guaranteed Value of β_{guar} of the Square of the Error of the Arithmetic Mean on the Number n of Measurements

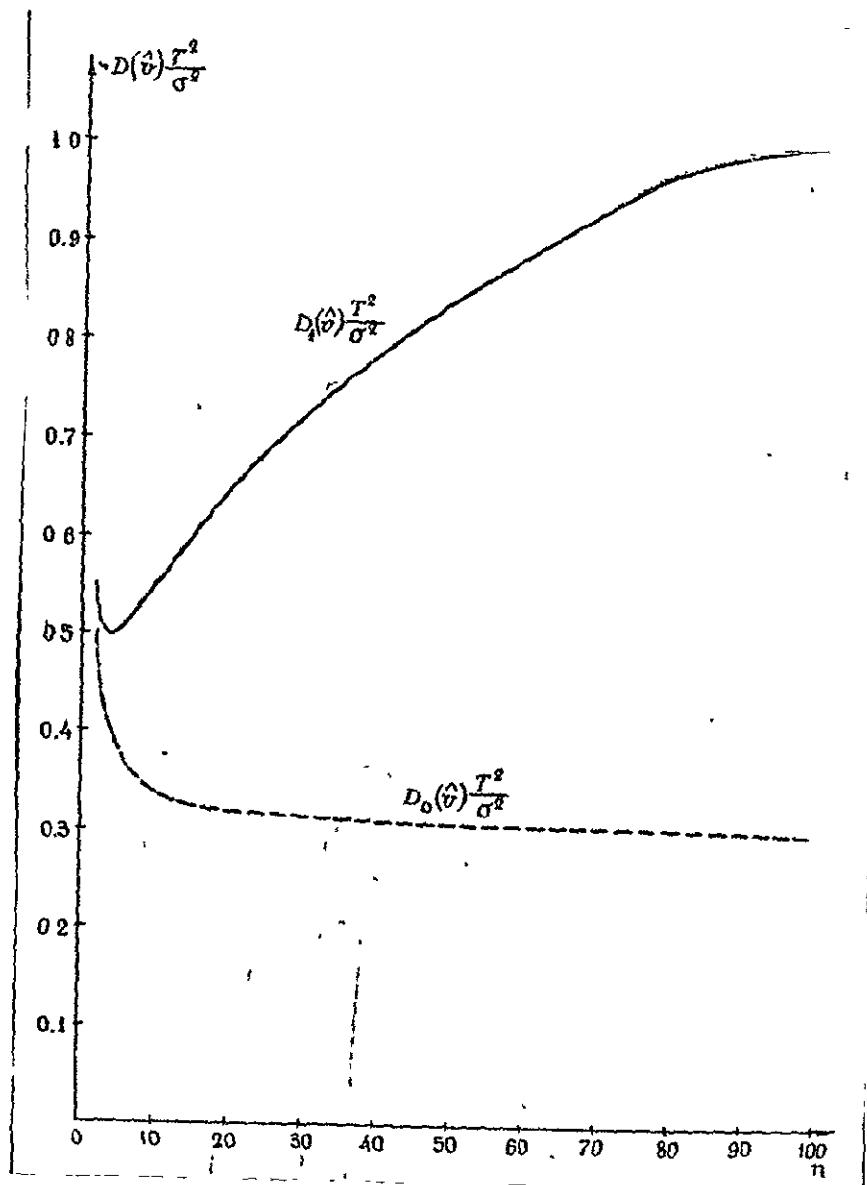


Figure 2
 Dependence of the Dispersion $D(\hat{v})$ of the Evaluation
 of the Velocity on the Number n of Measurements